Notion of integrability for time-dependent Hamiltonian systems: Illustrations from the relativistic motion of a charged particle

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It is shown that "Liouville's theorem" on integrability still holds in the case of time-dependent Hamiltonian systems; when they have n independent, possibly time-dependent, invariants the solution can be found with quadratures and no chaos can take place. This is applied to three important problems describing the motion of a particle in an electromagnetic field. The first is the motion of a charged particle in a homogeneous constant magnetic field and a transverse circularly polarized homogeneous electric field. In the second application the electric field is replaced by a standing electromagnetic wave. The third concerns an oscillator with a quadratic nonlinearity in the force. [S1063-651X(98)10801-2]

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I. INTRODUCTION

One might have expected that knowledge of $2n$ independent invariants would always be required to obtain the general solution for an autonomous Hamiltonian system with n degrees of freedom. This is suggested by the fact that Hamilton's equations are a set of $2n$ first-order equations. However, from a result due to Bour [1] and often attributed to Liouville [2], the existence of n independent, time-independent invariants in involution is sufficient to derive a general solution to the problem [3,4]. Liouville extended this result to nonautonomous systems by showing that n independent, possibly time-dependent, invariants in involution can be used to obtain n additional invariants [2,5]. However, he did not construct the $2n$ invariants as n canonically conjugate pairs [6].

An autonomous Hamiltonian system is called completely integrable if it possesses n independent, time-independent invariants in involution [3,4,7,8]. The solution for the motion of a completely integrable system can be expressed in terms of canonical action-angle variables [3,4,7,8]. Unfortunately, this does not mean that the equations of motion can be integrated analytically, but the solution always exists and is unique for specified initial data. Moreover, when the motion is completely integrable, all the Lyapunov exponents one can compute equal zero. Chaotic trajectories can fill only a phase-space volume of zero measure.

A general time-dependent Hamiltonian with n degrees of freedom is usually considered to be equivalent to an autonomous one in an extended phase space with $n+1$ degrees of freedom [8]. In Sec. II B we show that the equivalent system in the extended phase space is completely integrable if the original, nonautonomous system possesses n independent, possibly time-dependent, invariants in involution. The Hamiltonian in the extended phase space is introduced carefully and it can be proved that it is possible to construct $2(n+1)$ invariants as $n+1$ canonically conjugate pairs [6]. The demonstration for the case $n=1$ is given by one method in Sec. II A and by a second method in Sec. II B. On the other hand, it was shown by Kozlov and Kolesnikov that the solution of the original nonautonomous system can be found by quadratures [3,9]. This result has been shown explicitly for $n=1$, provided one invariant is specified, and a second invariant is then expressed in terms of quadratures, which permits one to derive formally the solution as a function of two constants [10–12]. This result is outlined Sec. II A. We have decided to call a time-dependent Hamiltonian system integrable if it possesses n independent, possibly time-dependent, invariants in involution. This is an extension of the first definition of integrability. In addition, it is shown that Lyapunov exponents are not positive in the case of integrable systems. Consequently, no chaos can take place. In this sense, we say that "Liouville's theorem" on integrability still holds in the case of time-dependent Hamiltonian systems.

We illustrate this concept for three examples in Sec. III. The first example concerns the study of the relativistic motion of a charged particle in a constant homogeneous magnetic field and transverse circularly polarized electric field [13–15]. It has been shown recently how the Hamiltonian formalism brings enlightenment to this problem [16]. The integrability of this two-degrees-of-freedom system first can be shown by finding two independent constants in involution, one of them obtained by using Noether's theorem [3,4,17,18]. Alternatively, the system can be reduced to a one-dimensional problem that can be solved by quadratures in two different ways and, because of the extension of the definition of integrability we have made, shown to be integrable [16]. Starting with the first approach, canonical transformations [3,4,7,8,16,18] permit one to take the two first invariants as two new conjugated variables and consequently we can reduce the system to a time-dependent Hamiltonian system with one degree of freedom possessing an invariant that is obtained by transforming the one derived from Noether's theorem. It is therefore integrable in the present "Liouville sense." As is shown in Sec. II A, a second integral can be derived that permits one to give the solution in terms of quadratures. Another approach by quadratures exists in which the solution can be written in terms of the energy of the particle and the energy is shown to be the solution of an integrable differential equation.

The second application deals with the dynamics of a charged particle in a constant homogeneous magnetic field and a transverse standing electromagnetic wave. This problem is more realistic than the former since the electromagnetic field that is considered satisfies Maxwell's equations exactly (in the first case, the field is only a solution in the limit of long wavelengths). Noether's theorem is still applied to derive an invariant. Canonical transformations are used in order to reduce the problem to a system with two degrees of freedom. No second invariant was found. Consequently, the problem may be nonintegrable and chaotic trajectories might exist in some circumstances. Other canonical transformations are used to change the problem into an autonomous one. After performing Poincaré maps and calculating positive Lyapunov exponents, we show that trajectories can be chaotic.

The third application is about the motion of a charged particle described by a radial equation in which anharmonicity is included with a term quadratic in position for the force. In addition, it is assumed that the anharmonicity depends on time, the coefficient of the non-linearity being a time-dependent function $f(t)$. It is shown that there exists at least one form for $f(t)$ such that this problem is integrable.

II. INTEGRABILITY OF TIME-DEPENDENT HAMILTONIAN SYSTEMS

In this section we study the integrability of an n-degrees-of-freedom time-dependent Hamiltonian system. However, in order to make the approach easier, we begin with $n=1$.

A. Integration by quadrature of one degree of freedom time-dependent systems

This is an outline of what has been done in Refs. [10–12]. We consider one-dimensional time-dependent problems where the Hamiltonian $H(Q,P,t)$ is a function of canonically conjugate variables (Q,P) and time t. It must be noted that H is not a constant of motion as it depends explicitly on time. The existence of a first integral $I(Q,P,t)$ is assumed. The formal inversion of I with respect to P gives

$$P=G(Q,I,t), \tag{1}$$

where G is the reciprocal function of I. Using Eq. (1) in one of Hamilton's equations, we get

$$\dot{Q}=h(Q,I,t), \tag{2}$$

where $h(Q,I,t)$ is a specified function. Equation (2) can be considered as a first-order differential equation with parameter I. Looking for an integrating factor $\alpha(Q,I,t)$ ($\alpha dQ-\alpha h dt$ is a total differential form), the partial derivative of the coefficient of dQ with respect to t must equal the partial derivative of the coefficient of dt with respect to Q, i.e., $\alpha_t=-(\alpha h)_Q$, or after development

$$\alpha_t+\alpha_Q h+\alpha h_Q=0, \tag{3}$$

where subscripts stand for the partial derivatives. One can show that a solution to Eq. (3) is [10,11]

$$\alpha=G_I(Q,I,t)=[I_P(Q,P,t)]^{-1}, \tag{4}$$

where $G_I=\partial G/\partial I$ and $I_P=\partial I/\partial P$. Then, to integrate Eq. (2), we note that since $G_I dQ-G_I h dt$ is a total differential, it can be interpreted as the total derivative of a first integral $J(Q,I,t)$ such that $dJ=G_I dQ-G_I h dt$ (no term is proportional to dI as I is a first integral). Under these circumstances, one must have

$$J_Q(Q,I,t)=G_I(Q,I,t), \quad J_t(Q,I,t)=-G_I(Q,I,t)h(Q,I,t). \tag{5}$$

Integration leads to

$$J(Q,I,t)=\int_0^Q \frac{dQ'}{I_P[Q';P=G(Q',I,t);t]} - \int_0^t \frac{H_P[0;P=G(0,I,t');t']}{I_P[0;P=G(0,I,t');t']}\,dt'. \tag{6}$$

In this way a second invariant is expressed in terms of quadratures [10–12]. As announced in the Introduction, I and J are canonically conjugate, i.e., $[J,I]=1$, where $[A,B]$ stands for the Poisson bracket of A with B [12]. Equation (6) shows that Q can be derived formally as a function of time and as a function of two arbitrary constants of motion I and J. This is in good agreement with the work of Kozlov and Kolesnikov, who proved that in such a case the solution can be found with quadratures [3,9]. In that sense our system is integrable.

B. Integrability of time-dependent systems with N degrees of freedom

Let us consider an autonomous Hamiltonian $\mathcal{H}$ $(q_j,p_j;\ j=1,n+1)$ with $n+1$ degrees of freedom of the specific form (the reason for this form will become clear later)

$$\mathcal{H}(q_j,p_j;j=1,n+1)=p_{n+1}+H(q_i,p_i,q_{n+1};i=1,n), \tag{7}$$

where H $(q_i,p_i,q_{n+1};\ i=1,n)$ is an arbitrary function of its arguments. Hamilton's equations are (τ is the time associated with this system)

$$\frac{dq_1}{d\tau}=\frac{\partial\mathcal{H}}{\partial p_1}=\frac{\partial H}{\partial p_1}, \tag{8a}$$

$$\frac{dp_1}{d\tau}=-\frac{\partial\mathcal{H}}{\partial q_1}=-\frac{\partial H}{\partial q_1}, \tag{8b}$$

$$\vdots$$

$$\frac{dq_j}{d\tau}=\frac{\partial\mathcal{H}}{\partial p_j}=\frac{\partial H}{\partial p_j}, \tag{8c}$$

$$\frac{dp_j}{d\tau}=-\frac{\partial\mathcal{H}}{\partial q_j}=-\frac{\partial H}{\partial q_j}, \tag{8d}$$

$$\vdots$$

$$\frac{dq_{n+1}}{d\tau}=\frac{\partial\mathcal{H}}{\partial p_{n+1}}=1, \tag{8e}$$

$$\frac{dp_{n+1}}{d\tau}=-\frac{\partial\mathcal{H}}{\partial q_{n+1}}=-\frac{\partial H}{\partial q_{n+1}}. \tag{8f}$$

Since this system is autonomous, $\mathcal{H}$ is conserved

$$\mathcal{H}=I_1, \tag{9}$$

where I_1 is a constant. Moreover, Eq. (8e) can be readily integrated to give

$$q_{n+1}-\tau=I_{n+1}, \tag{10}$$

where I_{n+1} is a constant. Thus two invariants are $\mathcal{H}$ and $q_{n+1}-\tau$ and we found them to be canonically conjugate

$$[q_{n+1}-\tau,\mathcal{H}]=1. \tag{11}$$

One concludes that, in the case $n=1$, two invariants can be derived for which the Poisson bracket is unity. This is in good agreement with Sec. II A.

Let us now replace the variable q_{n+1} by t (we will see below that t plays the role of the usual time) and one can write the equations of motion (8) as

$$\frac{dq_1}{dt}=\frac{\partial H(q_i,p_i,t)}{\partial p_1}, \tag{12a}$$

$$\frac{dp_1}{dt}=-\frac{\partial H(q_i,p_i,t)}{\partial q_1}, \tag{12b}$$

$$\vdots$$

$$\frac{dq_i}{dt}=\frac{\partial H(q_i,p_i,t)}{\partial p_i}, \tag{12c}$$

$$\frac{dp_i}{dt}=-\frac{\partial H(q_i,p_i,t)}{\partial q_i}, \tag{12d}$$

$$\vdots$$

$$\frac{dp_{n+1}}{dt}=-\frac{\partial H(q_i,p_i,t)}{\partial t}. \tag{12e}$$

The first $2n$ equations are those of an n-dimensional nonautonomous system for which $H(q_i,p_i,t)$ is the time-dependent Hamiltonian. Finally, since $dH/dt=\partial H/\partial t$, Eq. (12e) can be written as

$$\frac{dp_{n+1}}{dt}=-\frac{dH}{dt}. \tag{13}$$

By taking Eqs. (7) and (9) into account, its solution can be written as

$$p_{n+1}=-H+I_1. \tag{14}$$

This equation, which is a consequence of Eq. (12e), is just Eq. (7), where I_1 stands for $\mathcal{H}$. The difference between the autonomous Hamiltonian function $\mathcal{H}$ and its value I_1 must be emphasized. The value I_1 of $\mathcal{H}$ is an arbitrary constant along the solution trajectories, which can chosen to be zero. This result leads to an important remark concerning what has appeared in the literature about the extended phase space. When an n-dimensional time-dependent Hamiltonian $\widetilde{H}(q_i,p_i,t)$ $(i=1,n)$ is considered in Ref. [8], the authors set $q_{n+1}=t$ and $p_{n+1}=-\widetilde{H}$ and take $\widetilde{\mathcal{H}}(q_i,p_i,q_{n+1},p_{n+1})=\widetilde{H}+p_{n+1}\equiv 0$ as a Hamiltonian in the extended phase space. However, it is usually (incorrectly) omitted that this value is valid only along a solution trajectory. In addition, this is misleading as $\widetilde{\mathcal{H}}$ is a function identical to zero and cannot be considered as an arbitrary constant of motion. The function $\mathcal{H}$ can be chosen as a Hamiltonian and its value is an arbitrary constant if one sets

$$p_{n+1}=-\widetilde{H}+I_1. \tag{15}$$

Let us again consider Eqs. (12) and assume that $H(q_i,p_i,t)$ has n independent first integrals, possibly time dependent, $J_i(q_i,p_i,t)$ in involution. We have

$$\frac{dJ_i}{dt}=\frac{\partial J_i}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial J_i}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial J_i}{\partial p_i}\right)=0. \tag{16}$$

Then

$$[J_i,\mathcal{H}]=\sum_{i=1}^{n}\left(\frac{\partial J_i}{\partial q_i}\frac{\partial\mathcal{H}}{\partial p_i}-\frac{\partial J_i}{\partial p_i}\frac{\partial\mathcal{H}}{\partial p_i}\right)-\frac{\partial\mathcal{H}}{\partial t}\frac{\partial J_i}{\partial p_{n+1}}+\frac{\partial\mathcal{H}}{\partial p_{n+1}}\frac{\partial J_i}{\partial t}$$
$$=\sum_{i=1}^{n}\left(\frac{\partial J_i}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial J_i}{\partial p_i}\frac{\partial H}{\partial q_i}\right)+\frac{\partial J_i}{\partial t}\times 1=0. \tag{17}$$

Thus, because $\partial J_i/\partial\tau=0$ and $[J_i,\mathcal{H}]=0$, the integrals J_i are also invariants for the system associated with $\mathcal{H}(q_i,p_i,q_{n+1},p_{n+1})$. The system with $n+1$ degrees of freedom has $n+1$ independent invariants in involution and is therefore completely integrable ($\mathcal{H}$ is independent of the J_i's because they do not depend on p_{n+1}). Therefore, we have shown that the time-dependent system associated with H, which is integrable in the sense that the solution can be found by quadratures (according to Kozlov and Kolesnikov), is equivalent to a completely integrable system with one additional degree of freedom.

The solution of the system with $n+1$ degrees of freedom can be expressed in terms of canonical action-angle variables and, as a consequence, the solution for the system with n degrees of freedom can be found by using the inverse transformation. It should be noted that being able to transform the nonautonomous system to a completely integrable autonomous system does not imply that the solution of the nonautonomous system can be expressed simply. An example of that for a linear, one-degree-of-freedom, time-dependent system was given by Salat and Tataronis [19].

If the system is completely integrable in the extended phase space, the numerically calculated Lyapunov exponent corresponding to any trajectory cannot be positive. In this space one can consider pairs of trajectories with the same initial time τ and the same initial p_{n+1}. The Lyapunov exponents are given by $\sigma_{\mathrm{ext}}=\lim(1/\tau)\ln[D(\tau)/D(0)]$ in the limit when $\tau\to\infty$ and $D(0)\to 0$, where $D(\tau)$ measures the distance between the two solutions at a time τ [7,8]. In the same way, the corresponding Lyapunov exponents can be calculated in the original space from the limit $\sigma_{\rm orig} = \lim(1/t)\ln[d(t)/d(0)]$ when $t\to\infty$ and $d(0)\to 0$, where $d(t)$ represents the distance between the two solutions at time t. It is shown in the Appendix that $d(0)=D(0)$ and $d(t) \leqslant D(\tau)$. Then $\sigma_{\rm orig}\leqslant\sigma_{\rm ext}$ and one can conclude that any Lyapunov exponent cannot be positive in the original space provided none of them is positive in the extended phase space. As a consequence, in the case of time-dependent Hamiltonian problems with n independent constants of motion (possibly time dependent) in involution, only a set of measure zero can contain chaotic trajectories.

III. APPLICATIONS TO THE DYNAMICS OF A CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD

A. Integrability of the motion of a charged particle in a constant homogeneous magnetic field and a transverse rotating electric field

The constant magnetic field $\mathbf{B}_0$ is assumed to be along the z axis and the electric field has the components

$$E_x = E_0 \cos \omega_0 t, \quad E_y = E_0 \sin \omega_0 t, \quad E_z = 0, \tag{18}$$

where E_0 and ω_0 are constants. The following gauge is chosen for the electromagnetic field:

$$\mathbf{A} = -\left(\frac{B_0}{2}y + \frac{E_0}{\omega_0}\sin \omega_0 t\right)\hat{\mathbf{e}}_x + \left(\frac{B_0}{2}x + \frac{E_0}{\omega_0}\cos \omega_0 t\right)\hat{\mathbf{e}}_y. \tag{19}$$

As a consequence of Maxwell's equations, the electric field and the total magnetic field cannot be constant along the z axis. However, in this paper we consider long wavelengths and the variation of the magnetic field can be neglected in a region where the electric field is maximum. The motion of the charged particle is assumed to be in the x-y plane. Its relativistic Hamiltonian expressed in mks units is

$$H = \left[\left(P_x - \frac{eE_0}{\omega_0}\sin \omega_0 t - \frac{eB_0}{2}y\right)^2 c^2 + \left(P_y + \frac{eE_0}{\omega_0}\cos \omega_0 t + \frac{eB_0}{2}x\right)^2 c^2 + m^2c^4\right]^{1/2}, \tag{20}$$

where $-e$ and m are the charge and rest mass of the particle.

1. First demonstration of the integrability of the problem

Hamilton's equations allow us to find immediately two constants of motion [16]

$$C_1 = P_x + \frac{eB_0}{2}y, \quad C_2 = P_y - \frac{eB_0}{2}x. \tag{21}$$

Another constant of motion can be obtained by using Noether's theorem [3,4,16,17]. The result is [16]

$$C_3 = yP_x - xP_y + H/\omega_0. \tag{22}$$

It can be noted that the first two constants are canonically conjugate

$$\left[C_1, \frac{C_2}{eB_0}\right] = 1, \tag{23}$$

where now C_1 and C_2 must be considered as functions of their arguments [see Eqs. (21)]. Among these three constants of motion, one cannot find two of them in involution since

$$[C_1, C_3] = C_2, \quad [C_2, C_3] = -C_1. \tag{24}$$

However, another constant of motion is given by

$$C_4 = C_1^2 + C_2^2. \tag{25}$$

It satisfies the relation

$$[C_4, C_3] = 2C_1[C_1, C_3] + 2C_2[C_2, C_3] = 0. \tag{26}$$

As C_3 and C_4 are two independent constants in involution, the system is integrable according to the definition in the introduction.

2. Reduction to a one-dimensional problem, second demonstration of the integrability of the problem

What follows is a part of what has been done in Ref. [16]. Written in terms of the dimensionless variables and parameters

$$\hat{x} = x\frac{\omega_0}{c}, \quad \hat{y} = y\frac{\omega_0}{c}, \quad \hat{P}_{x,y} = \frac{P_{x,y}}{mc}, \quad \hat{t} = \omega_0 t,$$

$$\hat{H} = \gamma = \frac{H}{mc^2}, \quad a = \frac{eE_0}{mc\omega_0}, \quad \Omega_0 = \frac{eB_0}{m\omega_0},$$

the Hamiltonian is

$$\hat{H} = \left[\left(\hat{P}_x - a \sin \hat{t} - \frac{\Omega_0}{2}\hat{y}\right)^2 + \left(\hat{P}_y + a \cos \hat{t} + \frac{\Omega_0}{2}\hat{x}\right)^2 + 1\right]^{1/2}. \tag{27}$$

In these variables, the constants of motion corresponding to C_1 and C_2 are

$$\hat{C}_1 = \hat{P}_x + \frac{\Omega_0}{2}\hat{y}, \quad \hat{C}_2 = \hat{P}_y - \frac{\Omega_0}{2}\hat{x}. \tag{28}$$

They satisfy

$$[\hat{C}_1, \hat{C}_2] = \Omega_0. \tag{29}$$

By using this property we can show that the system can be described by a single-degree-of-freedom time-dependent Hamiltonian. We take $\hat{C}_1$ and $\hat{C}_2$ (one must be normalized by Ω_0) as new conjugate momentum and coordinate and we choose a canonical transformation $(\hat{x},\hat{y},\hat{P}_x,\hat{P}_y) \to (\tilde{x},\tilde{y},\tilde{P}_x,\tilde{P}_y)$ defined by the type-2 generating function [3,4,7,8,16]

$$F_2=\left(\tilde{P}_x-\frac{\Omega_0}{2}\hat{y}\right)\hat{x}+\tilde{P}_y\hat{y}. \qquad (30)$$

The canonical transformation is

$$\hat{x}=\tilde{x}, \quad \hat{y}=\tilde{y}, \quad \hat{P}_x=\tilde{P}_x-\frac{\Omega_0}{2}\tilde{y}, \quad \hat{P}_y=\tilde{P}_y-\frac{\Omega_0}{2}\tilde{x}. \qquad (31)$$

In these variables $\hat{C}_1$ and $\hat{C}_2$ become

$$\tilde{C}_1=\tilde{P}_x, \quad \tilde{C}_2=\tilde{P}_y-\Omega_0\tilde{x}. \qquad (32)$$

Then we introduce a second canonical transformation $(\tilde{x},\tilde{y},\tilde{P}_x,\tilde{P}_y)\rightarrow(Q_1,Q_2,P_1,P_2)$, generated by

$$F_2=(P_2+\Omega_0\tilde{x})\tilde{y}+P_1\left(\tilde{x}+\frac{P_2}{\Omega_0}\right) \qquad (33)$$

and yielding

$$\tilde{x}=Q_1-\frac{P_2}{\Omega_0}, \quad \tilde{y}=Q_2-\frac{P_1}{\Omega_0}, \quad \tilde{P}_x=\Omega_0 Q_2, \quad \tilde{P}_y=\Omega_0 Q_1. \qquad (34)$$

The resulting transformation, which is the product of the two transformations, is given by

$$\hat{x}=Q_1-\frac{P_2}{\Omega_0}, \quad \hat{y}=Q_2-\frac{P_1}{\Omega_0}, \quad \hat{P}_x=\frac{1}{2}(\Omega_0 Q_2+P_1),$$

$$\hat{P}_y=\frac{1}{2}(\Omega_0 Q_1+P_2). \qquad (35)$$

In terms of these variables one has

$$Q_2=\frac{\hat{C}_1}{\Omega_0}, \quad P_2=\hat{C}_2 \qquad (36)$$

and the Hamiltonian is

$$\overline{H}=[(P_1-a\sin\hat{t})^2+(\Omega_0 Q_1+a\cos\hat{t})^2+1]^{1/2}. \qquad (37)$$

As expected, P_2 and Q_2 are cyclic variables since they do not appear in the Hamiltonian. It depends upon time and has only one degree of freedom. In terms of these new variables, the constant C_3, which we now denote by K, is

$$K=\overline{H}-\frac{P_1^2}{2\Omega_0}-\frac{\Omega_0}{2}Q_1^2. \qquad (38)$$

The fact that this problem can be reduced to a one-degree-of-freedom system can help to predict the appearance of the trajectories easily [16]. As we have one constant of motion, we can conclude, according to the Introduction, that the system is integrable. This is a second way to show the integrability of Hamiltonian (20). Finally, according to Sec. II A, we can also conclude that a second constant of motion of Hamiltonian (37) can be obtained and the solution can be given in terms of quadratures.

3. Another way to solve the problem

The equations of Hamilton derived from Eq. (37) are

$$\dot{P}_1=-\frac{\Omega_0}{\gamma}(\Omega_0 Q_1+a\cos\hat{t}), \quad \dot{Q}_1=\frac{1}{\gamma}(P_1-a\sin\hat{t}). \qquad (39)$$

Introducing the variables

$$\overline{Q_1}=Q_1+a/\Omega_0\cos\hat{t}, \quad \overline{P_1}=P_1-a\sin\hat{t}, \qquad (40)$$

the complex quantity $Z=\overline{P_1}+i\Omega_0\overline{Q_1}$, taking into account that $\overline{H}=\gamma=\sqrt{1+|Z|^2}$, Hamilton's equations (39) are equivalent to

$$\dot{Z}=\frac{i\Omega_0 Z}{\sqrt{1+|Z|^2}}-a\exp(i\hat{t}), \qquad (41)$$

which is the equation of a nonlinear oscillator under the action of an external force. Remembering that $\gamma=\sqrt{1+|Z|^2}$, we can consider Eq. (41) as formally linear. Thus the solution of this equation can be written as

$$Z=A_0\exp i[\sigma(\hat{t})+\delta]-a\int_0^{\hat{t}}\exp i[\sigma(\hat{t})-\sigma(\tau)+\tau]d\tau, \qquad (42)$$

with

$$\sigma(\hat{t})=\Omega_0\int_0^{\hat{t}}d\tau\,\gamma^{-1}(\tau), \qquad (43)$$

where A_0 and δ are real constants. The solution is formal as it depends on γ, which is unknown. It will be used below to derive an equation for γ. Then

$$\begin{aligned} P_1&=A_0\cos[\sigma(\hat{t})+\delta]+a\sin\hat{t}\\ &\quad-a\int_0^{\hat{t}}\cos[\sigma(\hat{t})-\sigma(\tau)+\tau]d\tau, \end{aligned} \qquad (44)$$

$$\begin{aligned} Q_1&=\frac{A_0}{\Omega_0}\sin[\sigma(\hat{t})+\delta]-\frac{a}{\Omega_0}\cos\hat{t}-\frac{a}{\Omega_0}\\ &\quad\times\int_0^{\hat{t}}\sin[\sigma(\hat{t})-\sigma(\tau)+\tau]d\tau. \end{aligned}$$

The quantities A_0 and δ are determined so that at $\hat{t}=0$, $A_0^2=\gamma_0^2-1=\hat{p}_{x0}^2+\hat{p}_{y0}^2$ and $\tan\delta=\hat{p}_{y0}/\hat{p}_{x0}$ ($\hat{p}=p/mc$, p is the momentum of the particle). The subscript 0 appended to variables γ and p refers to their initial values.

We now derive an equation for γ. Taking the time derivative of Eq. (37) with respect to time and using Eqs. (44), we obtain

$$\begin{aligned} \gamma\dot{\gamma}=-a\Bigg\{&A_0\cos[\sigma(\hat{t})-\hat{t}+\delta]\\ &-a\int_0^{\hat{t}}\cos[\sigma(\hat{t})-\sigma(\tau)+\tau-\hat{t}]d\tau\Bigg\}. \end{aligned} \qquad (45)$$

This equation, multiplied by $-1+\Omega_0/\gamma$ and integrated between 0 and $\hat{t}$, leads to

$$\Omega_0(\gamma-\gamma_0)-\frac{\gamma^2-\gamma_0^2}{2}-aA_0\sin\delta$$
$$=-a\left\{A_0\sin[\sigma(\hat{t})-\hat{t}+\delta]-a\int_0^{\hat{t}}\sin[\sigma(\hat{t})-\sigma(\tau)+\tau-\hat{t}]d\tau\right\}. \quad (46)$$

Then Eq. (45) is differentiated with respect to time and Eq. (46), multiplied by $-1+\Omega_0/\gamma$, is added to it. The result is multiplied by $\gamma\dot{\gamma}$ and integrated between 0 and $\hat{t}$. In this way, the following differential equation for the energy is derived:

$$(\dot{\gamma})^2+\frac{\gamma^2}{4}-\Omega_0\gamma+R_0-\frac{\Gamma_0}{\gamma}-\frac{K_0}{\gamma^2}=0, \quad (47)$$

with

$$R_0=aA_0\sin\delta+\Omega_0^2+\Omega_0\gamma_0-\frac{\gamma_0^2}{2}-a^2, \quad (48)$$

$$\Gamma_0=2\Omega_0 aA_0\sin\delta+2\Omega_0^2\gamma_0-\Omega_0\gamma_0^2, \quad (49)$$

and

$$K_0=a^2A_0^2\cos^2\delta-\Gamma_0\gamma_0+R_0\gamma_0^2-\Omega_0\gamma_0^3+\frac{\gamma_0^4}{4}. \quad (50)$$

This result is in agreement with the one given by Roberts and Buchsbaum [13]. Equation (47) describes a motion in a one-dimensional potential. It admits a solution that gives time in terms of a sum of elliptic integrals of the first and third types [20]. Hence the fact that an analytical solution of Eq. (47) exists permits us, with the help of Eqs. (44), to express Q_1 and P_1 in terms of quadratures.

B. Integrability of the motion of a charged particle in a constant homogeneous magnetic field and a transverse circularly polarized standing wave

The constant magnetic field $\mathbf{B}_0$ is still supposed to be along the z axis (Fig. 1). The electromagnetic field now has the form

$$E_x=\widetilde{E}_0\cos(\omega_0 t)\sin(k_0 z),$$

$$E_y=\widetilde{E}_0\sin(\omega_0 t)\cos(k_0 z), \quad E_z=0,$$

$$B_x=\frac{k_0\widetilde{E}_0}{\omega_0}\cos(\omega_0 t)\sin(k_0 z),$$

$$B_y=-\frac{k_0\widetilde{E}_0}{\omega_0}\sin(\omega_0 t)\cos(k_0 z), \quad B_z=B_0 \quad (51)$$

and the vector potential for the electromagnetic field is given by

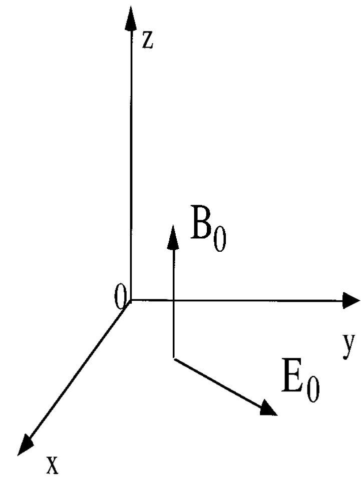

FIG. 1. Coordinate system.

$$\mathbf{A}=-\left(\frac{B_0}{2}y+\frac{\widetilde{E}_0}{\omega_0}\sin(\omega_0 t)\cos(k_0 z)\right)\hat{\mathbf{e}}_x+\left(\frac{B_0}{2}x+\frac{\widetilde{E}_0}{\omega_0}\cos(\omega_0 t)\cos(k_0 z)\right)\hat{\mathbf{e}}_y. \quad (52)$$

The relativistic Hamiltonian of the motion of one electron is

$$H=\left[\left(P_x-\frac{e\widetilde{E}_0}{\omega_0}\sin\omega_0 t\cos k_0 z-\frac{eB_0}{2}y\right)^2c^2+\left(P_y+\frac{e\widetilde{E}_0}{\omega_0}\cos\omega_0 t\cos k_0 z+\frac{eB_0}{2}x\right)^2c^2+P_z^2c^2+m^2c^4\right]^{1/2}. \quad (53)$$

This is a three-degrees-of-freedom time-dependent Hamiltonian. The invariants C_1 and C_2, which were defined in Sec. III A, still exist. Noether's theorem can be used and C_3 is still a constant of motion.

When $B_0=0$, the problem is described by a one degree-of-freedom Hamiltonian. Since we have at least one constant of motion $C_1=P_x$, $C_2=P_y$ or $C_3=yP_x-xP_y+H/w_0$), the system is therefore integrable. When $\widetilde{E}_0=0$, the Hamiltonian becomes autonomous and since H and P_z are two constants in involution, the problem is completely integrable. Now set $\hat{P}_z=P_z/mc$ and $\hat{z}=k_0 z$ and introduce the previous dimensionless variables and parameters (here E_0 is replaced by $\widetilde{E}_0$). In the case when $k_0c/\omega_0=1$, the following normalized Hamiltonian is obtained:

$$\hat{H}=\left[\left(\hat{P}_x-a\sin\hat{t}\cos\hat{z}-\frac{\Omega_0}{2}\hat{y}\right)^2+\left(\hat{P}_y+a\cos\hat{t}\cos\hat{z}+\frac{\Omega_0}{2}\hat{x}\right)^2+\hat{P}_z^2+1\right]^{1/2}. \quad (54)$$

Here again, the canonical transformation given by Eqs. (35) is used to reduce the problem to a time-dependent system with two degrees of freedom. The present Hamiltonian is

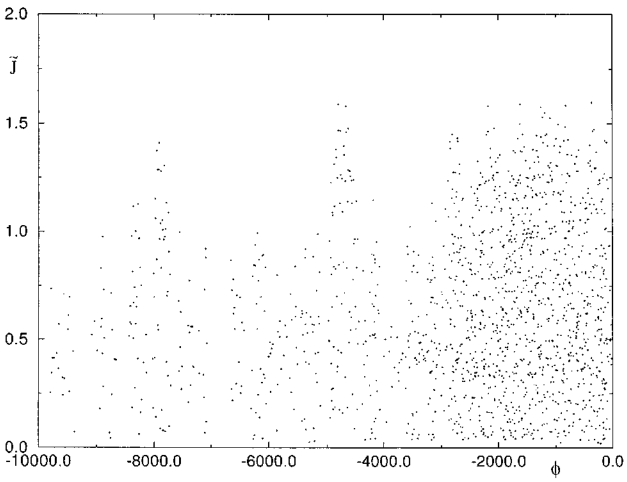

FIG. 2. Surface-of-section plots for some trajectories calculated with the equations of motion derived from Eq. (65) when $a=0.5$ and $\Omega_0=0.35$.

$$\overline{H}=[(p_1-a\ \sin\ \hat{t}\ \cos\ \hat{z})^2 + (\Omega_0 Q_1 + a\ \cos\ \hat{t}\ \cos\ \hat{z})^2 + \hat{P}_z^2+1]^{1/2}, \tag{55}$$

with the constant of motion

$$K=\overline{H}-\frac{P_1^2}{2\Omega_0}-\frac{\Omega_0}{2}Q_1^2. \tag{56}$$

We introduce action-angle variables for the Q_1,P_1 variables. The canonical transformation $(Q_1,P_1)\rightarrow(\theta,\tilde{J})$ is generated by

$$F_2(Q_1,\tilde{J})=\int_0^{Q_1}\sqrt{2(\tilde{J}\Omega_0-\tfrac{1}{2}\Omega_0^2Q_1^2)}\,dQ_1 \tag{57}$$

and yields

$$Q_1=\sqrt{2\tilde{J}/\Omega_0}\ \sin\ \theta, \quad P_1=\sqrt{2\tilde{J}\Omega_0}\ \cos\ \theta. \tag{58}$$

Then the Hamiltonian becomes

$$\bar{\bar{H}}=[2\tilde{J}\Omega_0+a^2\ \cos^2\ \hat{z}+2a\ \cos\ \hat{z}\ (2\tilde{J}\Omega_0)^{1/2} \times\sin(\theta-t)+\hat{P}_z^2+1]^{1/2} \tag{59}$$

and the first integral K becomes

$$\overline{K}=\bar{\bar{H}}-\tilde{J}. \tag{60}$$

Finally, another canonical transformation $(\theta,\tilde{J},\hat{z},\hat{P}_z)\rightarrow(\phi,\tilde{J},\hat{z},\hat{P}_z)$ is introduced, generated by $F_2(\theta,\hat{z},\tilde{J},P_z,\hat{t})=\tilde{J}(\theta-\hat{t})+\hat{z}\hat{P}_z$. It yields

$$\phi=\theta-\hat{t}. \tag{61}$$

In these variables, we have

$$\overline{\overline{\overline{H}}}=[2\tilde{J}\Omega_0+a^2\ \cos^2\ \hat{z}+2a\ \cos\ \hat{z}(2\tilde{J}\Omega_0)^{1/2} \times\sin\ \phi+\overline{P_z^2}+1]^{1/2}-\tilde{J}. \tag{62}$$

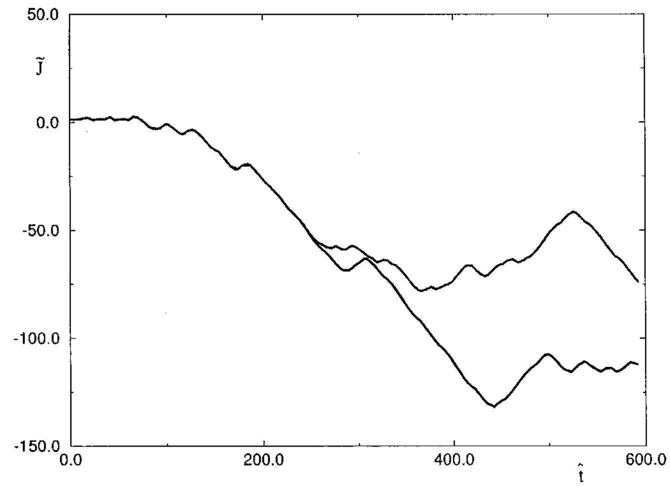

FIG. 3. Two trajectories calculated with the same initial conditions and two different time steps.

Since time is now ignorable in this Hamiltonian, it is a constant of motion and one can remark that with these variables the first integral obtained by using Noether's theorem is the Hamiltonian itself

$$\bar{\bar{K}}=\overline{\overline{\overline{H}}}, \tag{63}$$

where $\bar{\bar{K}}$ is obtained by transforming $\overline{K}$. Unfortunately, no other constant of motion has been found. Chaotic trajectories are evidenced by performing Poincaré maps. The plane $(\tilde{J},\phi)$ with $\hat{z}=0(\mathrm{mod}2\pi)$ is chosen to be the Poincaré surface of section (Fig. 2). Figure 3 shows two trajectories derived with the same initial conditions but with two different time steps. Although the energy is very well conserved in the two different cases, the two trajectories become rapidly different. This can be considered as a signature of chaos. Then a Poincaré map is performed for one of these trajectories and the points are indeed chaotic (Fig. 4). The Lyapunov exponent is also calculated for one of these trajectories. Benettin's method is used. This considers two trajectories with very close initial conditions. Renormalizations are performed every fixed time $\Delta\tau$ or each time the distance between the two

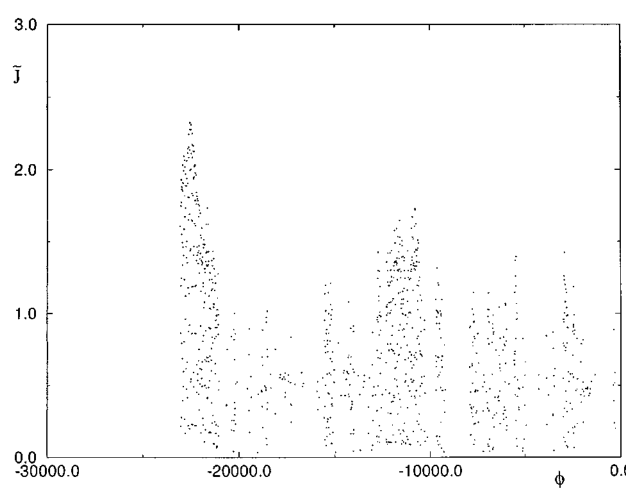

FIG. 4. Surface-of-section plots corresponding to one of the trajectories shown in Fig. 3.

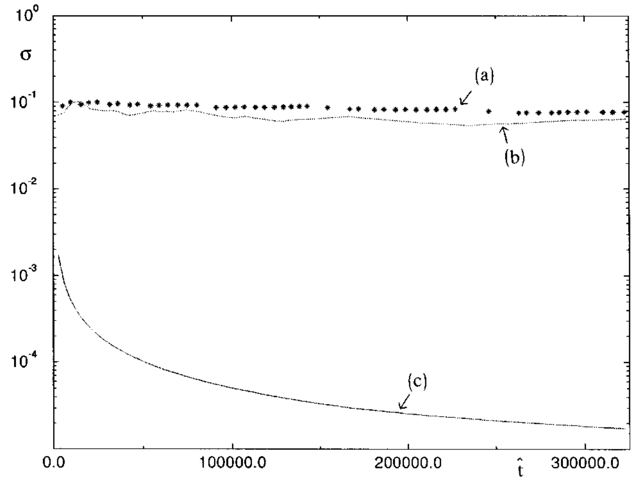

FIG. 5. Lyapunov exponents calculated for the same initial conditions as those of the trajectories shown in Fig. 3: (a) the renormalizations are performed every time the distance between the two trajectories is 2.7 times the initial distance, (b) renormalizations are performed every fixed time, and (c) in the integrable case when $a=0$ is considered, renormalizations are performed every fixed time.

trajectories is 2.7 times the distance between the initial conditions [21] (Fig. 5). This Lyapunov exponent is compared to the one obtained for the same initial conditions in a situation where the problem is known to be integrable (this occurs when there is a magnetic field only: $\widetilde{E}_0=0$ or $a=0$) (Fig. 5). The existence of chaotic trajectories proves that this system is not integrable. What we have just done is equivalent to applying the present Liouville's theorem to the system defined by Eq. (62); K is the only constant and no other one independent of K and in involution with it can be derived.

C. Integrability of time-dependent anharmonic oscillators with quadratic anharmonicity

The motion of a charged particle described by the equation of an harmonic oscillator perturbed by a force quadratic in the position is considered. The anharmonicity is assumed to depend explicitly on time through a coefficient $f(t)$. The equation is therefore

$$\ddot{q}+\omega^2 q+f(t)q^2=0, \tag{64}$$

where q is the radial position of the particle and ω the constant frequency of the field. This equation plays an important part in the field of the reversed-field pinch. It has already been extensively studied [22]. In this section, we look for a first integral, which permits one to show that this problem is integrable for a particular form of $f(t)$.

Equation (64) can be derived from the one-degree-of-freedom time-dependent Hamiltonian

$$H(q,p,t)=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2 q^2+\tfrac{1}{3}f(t)q^3, \tag{65}$$

where $p=dq/dt$. We seek a first integral $S(q,p,t)$ quadratic in the momentum and of the form [23]

$$S(q,p,t)=a_0(q,t)+a_1(q,t)p+a_2(q,t)p^2. \tag{66}$$

The arbitrary functions a_0, a_1, and a_2 can be determined by using the equation

$$\frac{dS}{dt}=\frac{\partial S}{\partial t}+p\,\frac{\partial S}{\partial q}-[\omega^2 q+f(t)q^2]\,\frac{\partial S}{\partial p}=0. \tag{67}$$

Using Eq. (66) for $S(q,p,t)$ in Eq. (67) gives, for the different powers of p,

$$\frac{\partial a_2}{\partial q}=0, \tag{68a}$$

$$\frac{\partial a_1}{\partial q}+\frac{\partial a_2}{\partial t}=0, \tag{68b}$$

$$\frac{\partial a_0}{\partial q}+\frac{\partial a_1}{\partial t}-2a_2[\omega^2 q+f(t)q^2]=0, \tag{68c}$$

$$\frac{\partial a_0}{\partial t}-a_1[\omega^2 q+f(t)q^2]=0. \tag{68d}$$

One has four equations for three unknown quantities. The system is therefore overdetermined and $S(q,p,t)$ defined by Eq. (66) can be a constant of motion only for a restricted set of functions $f(t)$. A consequence of Eq. (68a) is that

$$a_2=\alpha_2(t), \tag{69}$$

where $\alpha_2(t)$ is some time-dependent function. Inserting this form for a_2 into Eq. (68b) we find that

$$a_1(q,t)=-\dot{\alpha}_2 q+\alpha_1(t), \tag{70}$$

where $\alpha_1(t)$ is another time-dependent function. Then Eq. (68c) leads to

$$a_0(q,t)=\alpha_2\omega^2 q^2+\tfrac{2}{3}\alpha_2 f(t)q^3+\frac{\ddot{\alpha}_2}{2}\,q^2-\dot{\alpha}_1 q+\alpha_0(t). \tag{71}$$

Finally, using Eqs. (70) and (71) in Eq. (68d) and equating to zero all the coefficients of the powers of q, we obtain

$$\dot{\alpha}_0=0, \tag{72a}$$

$$\ddot{\alpha}_1+\alpha_1\omega=0, \tag{72b}$$

$$\frac{d^3\alpha_2}{dt^3}+4\omega^2\dot{\alpha}_2-2\alpha_1 f(t)=0, \tag{72c}$$

$$\frac{d(\alpha_2 f\,)}{dt}+\frac{3}{2}\,\dot{\alpha}_2 f(t)=0. \tag{72d}$$

As a consequence,

$$\alpha_2=\Lambda_0,\quad \alpha_1(t)=\Lambda_1\cos\omega t+\Lambda_2\sin\omega t, \tag{73}$$

where Λ_0, Λ_1, and Λ_2 are three arbitrary constants. Considering $\Lambda_1=\Lambda_2=0$ in Eq. (73), Eq. (72c) leads to

$$\alpha_2(t)=K_1+K_2\cos 2\omega t+K_3\sin 2\omega t, \tag{74}$$

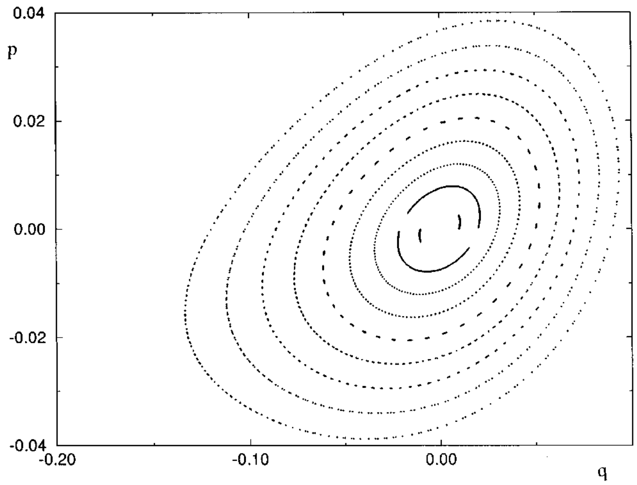

FIG. 6. Surface-of-section plots for some trajectories when $\dot{q}(0)=0$. The function $f(t)$ is given by Eq. (75) with $K_1=1$, $K_2=K_3=0.3$, and $\omega=0.5\ \mathrm{s}^{-1}$.

where K_1, K_2, and K_3 are three arbitrary constants. The solution of Eq. (72d) with Eq. (74) gives

$$f(t)=[K_1+K_2 \cos 2\omega t+K_3 \sin 2\omega t]^{-5/2}. \tag{75}$$

One can conclude that when $f(t)$ is under the form defined by Eq. (75), there is a constant of motion, as all the functions a_0, a_1, and a_2 entering S [Eq. (66)] are determined. As a consequence, this one-degree-of-freedom system is integrable. Poincaré maps were performed drawing one point at each period of $f(t)$ defined by Eq. (75) (Fig. 6). The distribution of the points is regular and this is in good agreement with the fact that the problem has a first integral and is integrable.

IV. CONCLUSIONS

The theoretical part of this paper (Sec. II) is an attempt to define the vague notion of integrability for time-dependent Hamiltonian systems with n $(n>1)$ degrees of freedom. Even in the case of autonomous Hamiltonian systems this notion covers several definitions (complete integrability [3,4,7,8], integrability by quadratures [6,9–12] and Painlevé integrability [24–26]). It is clear that the problem is more difficult in the case of nonautonomous systems. The difficulty arises in the way to transform back the properties found in the extended phase space to the initial nonautonomous phase space. For this reason, in this paper the Hamiltonian in the extended phase space has been carefully introduced in the case of a time-dependent problem with n degrees of freedom. What follows summarizes our main results. In this extended phase space, when one has n independent, possibly time-dependent, constants of motion in involution in the original space, one has $n+1$ independent invariants in involution. This permits us to conclude that, in this case, the system is completely integrable in the extended phase space. The n first integrals allow us to solve the system by quadratures and only a set of zero measure of the original space can be filled by chaotic trajectories. It is shown explicitly, in the case of a one-dimensional problem, that if one has one invariant, one can derive a second one that gives the solution in terms of quadratures. In this sense, we say that Liouville's theorem can be applied in its usual way. This generalized Liouville's theorem was applied to three problems concerning the motion of a charged particle.

First, the problem of relativistic motion of a charged particle in a constant homogeneous magnetic field and a transverse rotating electric field has been studied. The integrability of this problem was shown in two different ways. Noether's theorem was used to find a constant of motion for the system. Then a second integral was derived that is independent of and in involution with the first one. This is sufficient to prove the integrability as this problem has two degrees of freedom. Then, using canonical transformations, we reduced it to a time-dependent one with a single degree of freedom. This system has a constant of motion that is the one found previously by using Noether's theorem and expressed in the present variables. This was the second way to show that this problem is integrable. We have also shown, by using two different methods, that this system can be solved by quadratures. It was proved in Sec. II A that when a system has a constant of motion, the solution can be expressed in terms of quadratures. The second way consisted in deriving an equation for the energy which is integrable. As the coordinates were shown to be expressed under the form of quadratures containing the energy, the solution can be given in terms of quadratures.

The second application concerns the motion of an electron in a constant homogeneous magnetic field and a transverse standing electromagnetic wave. The system was reduced to a two-degrees-of-freedom problem. The ''nonintegrability'' was proved by performing Poincaré sections and calculating nonzero Lyapunov exponents.

The third application concerns the motion of a charged particle, which is described by the harmonic-oscillator equation perturbed by a quadratic term proportional to a time-dependent function $f(t)$. Deriving a first integral quadratic in the momentum and using the present Liouville theorem, it was shown that the problem is integrable for a certain class of functions $f(t)$.

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APPENDIX

It is shown in this appendix that the distance $d(t)$ between two trajectories with close initial conditions is smaller than the corresponding distance $D(\tau)$ in the extended phase space. The proof is given for a two-dimensional extended phase space, but it can be generalized easily to the case when it has n degrees of freedom. In the case of two degrees of freedom, Eq. (7) becomes

$$\mathcal{H}(q_1,p_1,q_2,p_2)=p_2+H(q_1,p_1,q_2)=\text{const}=k. \tag{A1}$$

In this space the flow is parametrized by the time τ. We consider two close trajectories 1 and 2 and we assume that the two initial times τ_1^i and τ_2^i satisfy

$$\tau_1^i = \tau_2^i = 0 \tag{A2}$$

and, consequently, the two trajectories have the same time $\tau = \tau_1 = \tau_2$.

The initial conditions can be chosen arbitrarily on each trajectory

$$q_1^1(0) = q_1^{i1}, \quad q_2^1(0) = q_2^{i1}, \quad p_1^1(0) = p_1^{i1}, \quad p_2^1(0) = p_2^{i1}, \tag{A3}$$

$$q_1^2(0) = q_1^{i2}, \quad q_2^2(0) = q_2^{i2}, \quad p_1^2(0) = p_1^{i2}, \quad p_2^2(0) = p_2^{i2}, \tag{A4}$$

where the superscripts 1 and 2 stand for the number of the trajectory and i means initial. On each trajectory ($j = 1,2$) the Hamiltonian $\mathcal{H}$ is a constant and its value k_j is obtained by introducing Eqs. (A3) and (A4) into Eq. (A1). We decide to set to zero the initial conditions

$$q_2^1(0) = q_2^2(0) = 0, \quad p_2^1(0) = p_2^2(0) = 0, \tag{A5}$$

but the quantities $q_1^1(0)$, $q_1^2(0)$, $p_1^1(0)$, and $p_1^2(0)$ remain arbitrary. Writing Hamilton's equations in the extended phase space, we find that $dp_2^1/d\tau$ and $dp_2^2/d\tau$ do not have the same values at the initial time $\tau = 0$. It turns out, therefore, that the further evolutions of p_2^1 and p_2^2 will differ and at any time τ we shall have

$$p_2^1(\tau) \neq p_2^2(\tau). \tag{A6}$$

Let us examine now what happens to $q_2^1(\tau)$ and $q_2^2(\tau)$. Hamilton's equations give

$$\frac{dq_2^1}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_2^1} = 1, \quad \frac{dq_2^2}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_2^2} = 1, \tag{A7}$$

which lead to the obvious solutions

$$q_2^1(\tau) = \tau + K_1, \quad q_2^2(\tau) = \tau + K_2, \tag{A8}$$

where K_1 and K_2 are two arbitrary constants. However, according to Eq. (A5), we must have $K_1 = K_2 = 0$ and one concludes that

$$q_2^1(\tau) = q_2^2(\tau) = \tau. \tag{A9}$$

The initial distance $D(\tau = 0)$ between the two trajectories is

$$D^2(\tau = 0) = [q_1^2(0) - q_1^1(0)]^2 + [p_1^2(0) - p_1^1(0)]^2 \tag{A10}$$

and at a time τ, taking into consideration Eq. (A9), it is

$$\begin{aligned} D^2(\tau) &= [q_1^2(\tau) - q_1^1(\tau)]^2 + [q_2^2(\tau) - q_2^1(\tau)]^2 \\ &\quad + [p_1^2(\tau) - p_1^1(\tau)]^2 + [p_2^2(\tau) - p_2^1(\tau)]^2 \\ &= [q_1^2(\tau) - q_1^1(\tau)]^2 + [p_1^2(\tau) - p_1^1(\tau)]^2 \\ &\quad + [p_2^2(\tau) - p_2^1(\tau)]^2. \end{aligned} \tag{A11}$$

Before returning to the initial space we make the following remarks. According to Eq. (10) (Sec. II B), one gets ($n = 1$)

$$q_2 = \tau + I_2. \tag{A12}$$

Moreover, by definition, we have

$$q_2 = t \tag{A13}$$

and, according to Eq. (A9), we conclude that on each trajectory $I_{2,1} = I_{2,2} = 0$. Consequently,

$$t = \tau. \tag{A14}$$

One can point out that, obviously, $\tau = 0$ corresponds to $t = 0$. Therefore, it follows from Eqs. (A3) and (A4) that

$$q_1(t=0) = q_1(\tau=0) = \mathcal{A}, \tag{A15}$$

$$p_1(t=0) = p_1(\tau=0) = \mathcal{B}, \tag{A16}$$

where $\mathcal{A}$ and $\mathcal{B}$ denote arbitrary values. On the other hand, the equation

$$\frac{dq_1}{dt} = \frac{\partial}{\partial p_1} H(q_1, p_1, t) \tag{A17}$$

is the same as

$$\frac{dq_1}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_1} = \frac{\partial}{\partial p_1} H(q_1, p_1, q_2). \tag{A18}$$

We conclude that $q_1(t) = q_1(\tau)$ and, in the same way, $q_2(t) = q_2(\tau)$.

Let us give the initial distance $d(t=0)$ in the initial space. We have

$$d^2(t=0) = [q_1^2(0) - q_1^1(0)]^2 + [p_1^2(0) - p_1^1(0)]^2 \tag{A19}$$

and we deduce that $D(\tau=0) = d(t=0)$. Moreover, at time t, we have $d^2(t) = [q_1^2(t) - q_1^1(t)]^2 + [p_1^2(t) - p_1^1(t)]^2$. Finally, since $q_1^1(t) = q_1^1(\tau)$, $q_1^2(t) = q_1^2(\tau)$, $p_1^1(t) = p_1^1(\tau)$, and $p_1^2(t) = p_1^2(\tau)$, it is obvious that $D^2(\tau) = d^2(t) + [p_2^2(\tau) - p_2^1(\tau)]^2$ and we obtain the inequalities

$$d^2(t) \leqslant D^2(\tau), \tag{A20}$$

$$\sigma_{\text{orig}} \leqslant \sigma_{\text{ext}}. \tag{A21}$$

As the system is completely integrable in the extended phase space, σ_{ext} cannot be positive, and the same is for σ_{orig}. As a consequence, there is no chaos in the initial time-dependent system if no chaos arises in the extended phase space.

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